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Model Answer M.A./M. Sc. (First Semester)

Differential Gemetry-I **Paper Code: AS-2220**

1.(a). The locus of points whose co-ordinates satisfy an equation $f(x^1, x^2, x^3) = 0$, is called a surface.

Two independent equations

$$f_1(x^1, x^2, x^3) = 0, \quad f_2(x^1, x^2, x^3) = 0,$$

define a curve in three dimensional space.

(b). The relations

$$t'^i = \kappa p^i, \quad p'^i = \tau b^i - \kappa t^i, \quad b'^i = -\tau p^i,$$

are known as Serret-Frenet formulae. t^i, p^i and b^i are unit tangent, unit principal normal and binormal respectively.

(c). Given that

$$X = (a\cos\theta, a\sin\theta, a\cos 2\theta)$$

$$t^i = \frac{dX}{ds} = (-a\sin\theta, a\cos\theta; -2a\sin 2\theta)\frac{d\theta}{ds} \quad -(1)$$

taking modulus of both side, we have

$$\frac{d\theta}{ds} = \frac{1}{\sqrt{a^2 + 4a^2\sin^2 2\theta}} = \frac{1}{a\sqrt{1+4\sin^2 2\theta}}$$

At $\theta = \pi/6$

$$X = \left(\frac{a\sqrt{3}}{2},\ a/2,\ a/2 \right)$$

and $$t^i \equiv \left(-\frac{a}{2},\ \frac{a\sqrt{3}}{2},\ -\frac{2a\sqrt{3}}{2} \right) \cdot \frac{1}{\sqrt{1+4\times 3/4}}$$

$$t^i \equiv \left(-\frac{a}{4},\ \frac{a\sqrt{3}}{4},\ -\frac{a\sqrt{3}}{2} \right)$$

This is required unit tangent vector.

(d). In x-y plane

$$X = (x, y, 0)$$
$$X_1 = (1, 0, 0)$$
$$X_2 = (0, 1, 0)$$

Now
$$g_{11} = X_1 \cdot X_1 = 1$$
$$g_{12} = X_1 \cdot X_2 = 0$$
$$g_{22} = 1$$

Riemannian metric $$g = g_{11}\, g_{22} - g_{12}^2$$
$$= 1.$$

(e). The normal curvature of the surface in the direction of given curve C is defined as

$$K_n \overset{def}{=} \frac{d_{\alpha\beta}\, du^\alpha\, du^\beta}{g_{\alpha\beta}\, du^\alpha\, du^\beta},$$

K_n is curvature of normal sections.

(f). Given

$$\bar{r}(t) = (a\cos t,\ a\sin t,\ ct) \quad \text{——— (1)}$$

$$\frac{d\bar{r}}{dt} = (-a\sin t,\ a\cos t,\ c)$$

Now $$\frac{ds}{dt} = \sqrt{\frac{d\bar{r}}{dt}\cdot\frac{d\bar{r}}{dt}}$$

$$= \sqrt{a^2+c^2}$$

$$ds = \sqrt{a^2+c^2}\,dt$$

$$S = \int_{(a,0,0)}^{(a,0,2\pi c)} \sqrt{a^2+c^2}\,dt$$

$$= \int_{t=0}^{2\pi} \sqrt{a^2+c^2}\,dt \qquad \text{using (1)}$$

$$= 2\pi\sqrt{a^2+c^2}$$

this is required length.

(g). The osulating sphere or the sphere of curvature at point P on a curve is the sphere which has four-point contact with curve at any point P.

(h). $$X \equiv (a(u+v),\ b(u-v),\ uv)$$

$$X_1 \equiv (a,\ b,\ v)$$

$$X_2 \equiv (a,\ -b,\ u)$$

$$X_1 \times X_2 = \begin{vmatrix} i & j & k \\ a & b & v \\ a & -b & u \end{vmatrix} = i(u+v)b - j(u-v)a + k(-2ab)$$

$$\Rightarrow \quad X_1 \times X_2 \equiv \left(b(u+v),\ a(v-u),\ -2ab \right)$$

unit normal to the given surface is given by

$$\hat{N} = \frac{X_1 \times X_2}{|X_1 \times X_2|}$$

$$= \frac{\left(b(u+v),\ a(v-u),\ -2ab \right)}{\sqrt{b^2(u+v)^2 + a^2(v-u)^2 + 4a^2b^2}}$$

Ans.

(ii) $\quad X = (u, v, u^2 - v^2)$

$$X_1 = (1, 0, 2u), \quad X_{11} = (0, 0, 2)$$

$$X_2 = (0, 1, -2v), \quad X_{22} = (0, 0, -2)$$

$$X_{12} \equiv X_{21} = (0, 0, 0)$$

Now $\quad \hat{N} = \dfrac{X_1 \times X_2}{|X_1 \times X_2|}$

$$X_1 \times X_2 = \begin{vmatrix} i & j & k \\ 1 & 0 & 2u \\ 0 & 1 & -2v \end{vmatrix}$$

$$= (-2u, 2v, 1)$$

$$\hat{N}i = (-2u, 2v, 1)\frac{1}{\sqrt{1+4u^2+4v^2}}$$

$g_{11} = X_1 \cdot X_1 = 1 + 4u^2, \quad g_{22} = 1 + 4v^2$

$g_{12} = -4uv, \quad g = g_{11} g_{22} - g_{12}^2$

$d_{11} = X_{11} \cdot \hat{N}^i = 2/\sqrt{1+4u^2+4v^2}$

$d_{12} = X_{12} \cdot \hat{N}^i = 0$

$d_{12} = -2/\sqrt{1+4u^2+4v^2}$, Required metric.

(j). If K and K_n denote the curvature of oblique and normal sections through the same tangent line, and θ be the angle between the sections

$$K_n = K \cos\theta,$$

2(a). Given that

$$x^1 = 2a(\sin^{-1}t + t\sqrt{1-t^2}),$$

$$x^2 = 2at^2$$

$$x^3 = 4at$$

Now.

$$\dot{x}^1 = 2a\left(\frac{1}{\sqrt{1-t^2}} + \sqrt{1-t^2} - \frac{t^2}{\sqrt{1-t^2}}\right)$$

$$\dot{x}^2 = 4at$$

$$\dot{x}^3 = 4a$$

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx^1}{dt}\right)^2 + \left(\frac{dx^2}{dt}\right)^2 + \left(\frac{dx^3}{dt}\right)^2}$$

$$s = \int_{t_1}^{t_2} 4a\sqrt{2}\, dt = 4\sqrt{2}\,a\,(t_2 - t_1).$$

(b) Given curve is

$$\bar{r} = \left(t, \frac{1+t}{t}, \frac{1-t^2}{t}\right)$$

Above curve will lie in a plane if $\tau = 0$

i.e $[\dot{\bar{r}}\ \ddot{\bar{r}}\ \dddot{\bar{r}}] = 0.$

Now $\dot{\bar{r}} = \left(1, -\frac{1}{t^2}, -\frac{1}{t^2} - 1\right)$

$$\ddot{\bar{r}} = \left(0, \frac{2}{t^3}, 2/t^3\right)$$

$$\dddot{\bar{r}} = \left(0, -\frac{6}{t^4}, -\frac{6}{t^4}\right)$$

$$[\dot{\bar{r}}\ \ddot{\bar{r}}\ \dddot{\bar{r}}] = \begin{vmatrix} 1 & -\frac{1}{t^2} & -\frac{1}{t^2} - 1 \\ 0 & \frac{2}{t^3} & 2/t^3 \\ 0 & -6/t^4 & -6/t^4 \end{vmatrix}$$

$$= 0$$

Therefore given curve lies in a plane.

3(a). Given curve is

$$\bar{r} = (e^t, e^{-t}, \sqrt{2}\, t)$$

$$\Rightarrow \quad \dot{\bar{r}} = (e^t, -e^{-t}, \sqrt{2})$$

$$\ddot{\bar{r}} = (e^t, e^{-t}, 0)$$

$$\dddot{\bar{r}} = (e^t, -e^{-t}, 0)$$

$$\dot{\bar{r}} \times \ddot{\bar{r}} = \begin{vmatrix} i & j & k \\ e^t & -e^{-t} & \sqrt{2} \\ e^t & e^{-t} & 0 \end{vmatrix}$$

$$= (-\sqrt{2}\, e^{-t}, \sqrt{2}\, e^t, 2)$$

$$|\dot{\bar{r}} \times \ddot{\bar{r}}| = \sqrt{2e^{-2t} + 2e^{2t} + 4}$$

$$= \sqrt{2}\sqrt{e^{2t} + e^{-2t} + 2}$$

$$= \sqrt{2}\,(e^t + e^{-t})$$

and $$|\dot{\bar{r}}| = \sqrt{e^{2t} + e^{-2t} + 2}$$

$$= e^t + e^{-t}$$

$$K = \frac{|\dot{\bar{r}} \times \ddot{\bar{r}}|}{|\dot{\bar{r}}|^3} = \frac{\sqrt{2}\,(e^t + e^{-t})}{(e^t + e^{-t})^3}$$

$$\Rightarrow \quad \rho = \frac{(e^t + e^{-t})^2}{\sqrt{2}}$$

Now find p^i, τ and b^i and putting these values in equation

$$x^i = x^i + \rho \left\{ p^i + \cot\left(\int \tau\, ds + c \right) b^i \right\}$$

This is required equation of evolute of the given curve.

(b). Let $x^i = x^i(s)$ be a given curve C, with t^i, p^i and b^i as the tangent, principal normal and binormal of the curve C. The tangent surface C is given by

$$X^i = x^i + u t^i \qquad \text{———} \quad (I)$$

Putting $u = \phi(s)$, we get a curve on tangent surface. Then parametric equation of this curve $\bar{C}$ are

$$X^i = x^i + \phi(s) t^i \qquad \text{———} \quad (II)$$

Differentiating the equation (2) with respect to s, we get

$$X'^i = (1+\phi')t^i + \phi K p^i \qquad \text{———} \quad (III)$$

Since $\bar{C}$ is an involute, the vector X'^i should be orthogonal to the tangent vector of the given curve C, i.e

$$t^i . X'^i = 0$$

$$\Rightarrow \quad 1 + \phi' = 0$$

$$\Rightarrow \quad \phi = c - s \qquad \text{———} \quad (IV)$$

Hence equation of involute is given by

$$\boxed{X^i = x^i + (c-s)t^i} \qquad \text{From (II) and (IV).}$$

4. Given paraboloid is

$$2z = 5x^2 + 4xy + 2y^2 \qquad \text{——— (1)}$$

Now $X = \left(x, y, \dfrac{5x^2 + 4xy + 2y^2}{2} \right)$

$X_1 = (1, 0, 5x + 2y)$

$X_2 = (0, 1, 2y + 2x)$

$X_{11} = (0, 0, 5)$

$X_{22} = (0, 0, 2)$

$X_{12} = (0, 0, 2)$

$$X_1 \times X_2 = \begin{vmatrix} i & j & k \\ 1 & 0 & 5x+2y \\ 0 & 1 & 2y+2x \end{vmatrix}$$

at origin $x = y = 0$

$$X_1 \times X_2 = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}$$

$$= (0, 0, 1)$$

$N^{i} = (0, 0, 1)$

$g_{11} = X_1 \cdot X_1 = 1 + (5x + 2y)^2$ at origin

$= 1$

$g_{12} = X_1 \cdot X_2 = (5x + 2y)(2y + 2x)$ at origin

$= 0$

$$g_{22} = X_2 \cdot X_2$$
$$= 1 + (2y + 2x)^2$$
$$= 1 \qquad \text{at origin}$$

$$d_{11} = X_{11} \cdot N^i$$
$$= (0,0\,5) \cdot (0,0,1) = 5$$
$$d_{12} = (0,0,2)(0,0,1) = 2$$
$$d_{22} = (0,0,2)(0,0,1) = 2$$

Now normal curvature is given by

$$K_n = \frac{d_{11}(du^1)^2 + 2d_{12}\,du^1 du^2 + d_{22}(du^2)^2}{g_{11}(du^1)^2 + 2g_{12}(du^1)\,du^2 + g_{22}(du^2)^2}$$

$$= \frac{d_{11}(dx)^2 + 2d_{12}\,dx\,dy + d_{22}(dy)^2}{g_{11}(dx)^2 + 2g_{12}\,dx\,dy + g_{22}(dy)^2}$$

since $x = y \Rightarrow dx = dy$

$$K_n = \frac{d_{11} + 2d_{12} + d_{22}}{g_{11} + 2g_{12} + g_{22}}$$

$$= \frac{5 + 4 + 2}{1 + 0 + 1}$$

$$= \frac{11}{2}$$

Ans

5(a). Statement: "The sum of normal curvatures in two orthogonal directions is constant, and equal to sum of the principal curvatures".

Proff: By Euler's theorem we know that

$$K_{n_1} = K_1 \cos^2\alpha + K_2 \sin^2\alpha \quad \text{——— (I)}$$

and $$K_{n_2} = K_1 \cos^2\left(\frac{\pi}{2}+\alpha\right) + K_2 \sin^2\left(\frac{\pi}{2}+\alpha\right)$$

$$= K_1 \sin^2\alpha + K_2 \cos^2\alpha \quad \text{——— (II)}$$

Adding eqn (I) and (II), we have

$$K_{n_1} + K_{n_2} = K_1 + K_2$$

$$= \text{Constant}.$$

(b). Given that

$$\bar{r} = (a e^u \cos u, a e^u \sin u, b e^u)$$

$$s = \int_{-\infty}^{u} e^u (2a^2+b^2)^{1/2} \, du$$

$$= e^u (2a^2+b^2)^{1/2}$$

Find curvature and torsion in terms of s

i.e $$K = \frac{\sqrt{2}\, a}{(2a^2+b^2)^{1/2}} \cdot \frac{1}{s}, \quad \tau = \frac{b}{(2a^2+b^2)^{1/2}} \cdot \frac{1}{s}$$

Which are required equation (intrinsic) for given curve.

6. Equation of the helicoid is

$$x^3 = c \tan^{-1}\left(\frac{x^2}{x^1}\right) \quad \text{———} \quad (1)$$

Now parametric equation can be written as

$$x^1 = u \sin\theta, \quad x^2 = u \cos\theta, \quad x^3 = c\theta$$

$$\Rightarrow \quad X = (u\sin\theta, \ u\cos\theta, \ c\theta)$$

$$\Rightarrow \quad X_1 = (\sin\theta, \ \cos\theta, \ 0)$$

$$X_2 = (u\cos\theta, -u\sin\theta, \ c)$$

Evaluate K_1, K_2.

$$K_1 = c/(c^2+u^2), \quad K_2 = -c/u^2+c^2$$

Now, We know that

$$\text{Mean curvature} = \frac{K_1+K_2}{2}$$

$$= 0$$

Gaussian Curvature.

$$= K_1 \cdot K_2$$

$$= \frac{c^2}{(u^2+c^2)^2}$$

Ans.

7. From the definition of spherical indicatrix of tangent we have

$$\bar{r}_1 = t \qquad \text{———} \quad (I)$$

Differentiating w.r.t s, we get

$$\frac{dr_1}{ds_1} \cdot \frac{ds_1}{ds} = \kappa p$$

$$\Rightarrow \quad t_1 \frac{ds_1}{ds} = \kappa p \qquad \text{———} \quad (II)$$

$$\left| t_1 \frac{ds_1}{ds} \right| = |\kappa p|$$

$$\Rightarrow \quad \frac{ds_1}{ds} = \kappa \qquad \text{———} \quad (III)$$

From (II) and (III), we have

$$t_1 = p \qquad \text{———} \quad (IV)$$

Differentiating w.r.t s, we have

$$\kappa_1 \kappa n_1 = \tau b - \kappa t \qquad \text{———} \quad (V)$$

$$\Rightarrow \quad \kappa_1^2 = \frac{\kappa^2 + \tau^2}{\kappa^2}$$

taking cross product of (IV) and (V), we have

$$\kappa_1 \kappa b_1 = \tau t + \kappa b$$

Differentiating w.r.t s and taking dot product with (V) we have

$$\tau_1 = \frac{\kappa \tau' - \kappa' \tau}{\kappa(\kappa^2 + \tau^2)}.$$

8. Given

$$X = \left(x,\, y,\, \frac{ax^2 + 2hxy + by^2}{2}\right)$$

$$\Rightarrow \begin{cases} X_1 = (1,\, 0,\, ax + hy) \\ X_2 = (0,\, 1,\, by + hx) \\ X_{11} = (0,\, 0,\, a) \\ X_{12} = (0,\, 0,\, h) \\ X_{22} = (0,\, 0,\, b) \end{cases}$$

$$X_1 \times X_2 = \begin{vmatrix} i & j & k \\ 1 & 0 & ax + hy \\ 0 & 1 & by + hx \end{vmatrix}$$

$$= (-ax - hy,\, -by - hx,\, 1)$$

Now Normal

$$\hat{N} = \frac{X_1 \times X_2}{|X_1 \times X_2|}$$

$$= \frac{(-ax - hy,\, -by - hx,\, 1)}{\sqrt{1 + (ax + hy)^2 + (by + hx)^2}}$$

Riemannian metric

$$g_{11} = X_1 \cdot X_1 = 1 + (ax + hy)^2$$

$$g_{22} = X_2 \cdot X_2 = 1 + (by + hx)^2$$

$$g_{12} = X_1 \cdot X_2 = (ax + hy)(by + hx)$$

Find $g = g_{11}\, g_{22} - g_{12}^2$

Second fundamental magnitudes

$$d_{11} = X_{11} \cdot N^i$$
$$= \frac{a}{\sqrt{1 + (ax + hy)^2 + (by + hx)^2}}$$

$$d_{12} = X_{12} \cdot N^i$$
$$= h / \sqrt{1 + (ax + hy)^2 + (by + hx)^2}$$
$$= d_{21}$$

$$d_{22} = X_{22} \cdot N^i$$
$$= b / \sqrt{1 + (ax + hy)^2 + (by + hx)^2}$$

These are required fundamental magnitudes. Ans:

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